# Dirichlet distribution

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## 1 Derivation

Herewith, I derive mean of Dirichlet distribution (the derivation was shown in a YouTube Video[1]).

Let $x_k{}_{k=1}^{K}$ follows a Dirichlet distribution:

$$\begin{aligned} \boldsymbol{x} &\sim Dir(\boldsymbol{x}|\boldsymbol{\alpha}) \\ Dir(\boldsymbol{x}|\boldsymbol{\alpha}) &= \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}\prod_{k=1}^{K}x_k^{\alpha_k-1}. \end{aligned} \tag{1}$$

This is easily transformed into the following:

$$\int\prod_{k=1}^{K}x_k^{\alpha_k-1}d\boldsymbol{x} = \frac{\prod_{k=1}^{K}\Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{K}\alpha_k)}, \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_K)$.

Now, let's derive the mean of $x_k$. The mean of $x_k$, $E(x_k)$, is expressed as follows:

$$\begin{aligned} E(x_k) &= \int x_k Dir(\boldsymbol{x}|\boldsymbol{\alpha})d\boldsymbol{x} \\ &= \int x_k \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}\prod_{k=1}^{K}x_k^{\alpha_k-1}d\boldsymbol{x} \\ &= \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}\int x_k(\prod_{i=1,i\neq k}^{K}x_i^{\alpha_i-1})x_k^{\alpha_k-1}d\boldsymbol{x} \\ &= \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}\int(\prod_{i=1,i\neq k}^{K}x_i^{\alpha_i-1})x_k^{\alpha_k-1+1}d\boldsymbol{x}. \end{aligned} \tag{3}$$

Now the last integral can be re-written based on eq. (2).

$$\begin{aligned} \int(\prod_{i=1,i\neq k}^{K}x_i^{\alpha_i-1})x_k^{\alpha_k-1+1}d\boldsymbol{x} &= \frac{(\prod_{i=1,i\neq k}^{K}\Gamma(\alpha_i))\Gamma(\alpha_k+1)}{\Gamma(\sum_{k=1}^{K}\alpha_k+1)} \\ &= \frac{\alpha_k\prod_{k=1}^{K}\Gamma(\alpha_k)}{\sum_{k=1}^{K}\alpha_k\Gamma(\sum_{k=1}^{K}\alpha_k)}. \end{aligned} \tag{4}$$

It is easy to derive $E(x_k) = \alpha_k/\sum_{k=1}^{K}\alpha_k$, by using eqs. (3) and (4).

## References

[1] *Dirichlet Distribution: Mean* https://youtu.be/wrD1wI8etAI